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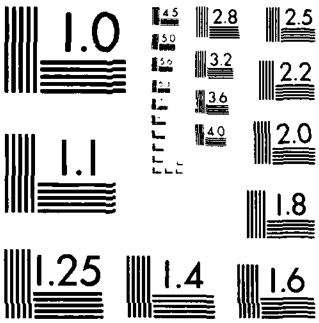

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Newton's Method as a Dynamical System: Global Convergence and Predictability

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Newton's method as an iterative scheme to compute both unstable and stable fixed points of a discrete dynamical system is considered. It is shown for Newton iterations that the basins of attraction are intertwined in a complicated manner. This complex structure appears to be fractal, and its dimension is estimated. Consequences of predictability for the final state are given in terms of imprecision in the initial data.

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NEWTON'S METHOD AS A DYNAMICAL SYSTEM: GLOBAL CONVERGENCE AND PREDICTABILITY

INTRODUCTION

When studying physical problems that exhibit chaotic behavior, it is known that unstable periodic orbits play a significant role in the analysis of the attractor (e.g.,[1,2,3]). Physical examples of the importance of the invariant unstable manifolds arising from unstable periodic orbits can be found in [4]. However, computing the location of unstable periodic orbits for either differential equations or maps cannot be successful without constructing a contracting iterative scheme due to the unstable nature of the orbit. Furthermore, it is even more difficult to use such a contracting scheme to prove rigorously the existence of an unstable periodic orbit.

One particular scheme that has been successful in computing unstable periodic orbits is that of Newton's Method (NM). See, for example, References [5,6]. Moreover, perturbations of NM have been used to give computer-assisted proofs of the Feigenbaum conjectures [7], the existence of periodic orbits of the Lorenz Equations [8,9,10], and the existence of unstable periodic orbits arising from saddle-node bifurcations in forced oscillators [6]. Newton's Method, whether applied to a differential equation in the form of finding fixed points of a Poincare map, or computing fixed points of a given discrete map, is itself an iterative procedure. Convergence properties of NM are almost always given in terms of the proximity of the iterates to the fixed point. That is, if one starts close enough to the fixed point, then under suitable hypotheses, the Newton iterates converge to the fixed point, where the number of significant digits doubles with every successive iterate.

Recently, the global behavior of Newton iterations has come under investigation in terms of dynamical systems and chaos [11,12,13]. In this paper, we consider how NM globally changes the structure of the basins of attraction for a map, T, which is bistable; i.e., the map has two stable fixed points, and the basins of attraction are intertwined in a complicated manner. Intuitively, it is expected that NM destroys the complicated basin structure due to the contracting nature of the iterates. It is not intuitively obvious that the Newton map will have its own unique and complex basin structure. Newton iterates for initial conditions close to a given fixed point are observed to converge to a completely different fixed point. In fact, initial conditions chosen arbitrarily close to each other are observed to converge to different fixed points. We show that the structure of the basins of attraction for a Newton

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map are very complicated. The complex structure of the basins leads to a definition of the predictability (following [14]) of the final state of Newton iterates as a function of precision in initial conditions. Predictability is given as a scaling law, from which the capacity [15] is estimated.

THE MAP T

We consider the two-dimensional iterates given by $y_{n+1} = T(y_n)$, where $y = (0, x)$, and

$$\begin{pmatrix} \theta \\ x \end{pmatrix} = \begin{pmatrix} \theta + A \sin 2\theta - B \sin 4\theta \; x \sin 4\theta \\ -J_0 \cos \theta \end{pmatrix}$$

The map T is periodic in θ with period 2π, and symmetric with respect to reflection about the line $\theta = \pi$.

The map T has five fixed points (i.e., those values $y^* = T(y^*)$) in the interval $0 \leqslant \theta \leqslant \pi$, whose stability characteristics are given in Table 1. The stability of each fixed point is determined by evaluating the eigenvalues of the Jacobian matrix of T at the fixed point under consideration. The fixed point is stable if all of the eigenvalues have modulus less than unity, and unstable if at least one eigenvalue has modulus greater than unity.

Table 1 — Fixed Points of Map T: Parameter Values:
$A = 1.64$, $J_0 = 0.3$ $B = 0.9$

Fixed Point (θ, x)	Eigenvalues of $T'(\theta, x)$
$(0, -J_0)$	$\lambda_+ = 0.98$ $\lambda_- = 0$
(θ, J_0)	$\lambda_+ = 0.98$ $\lambda_- = 0$
$(\theta/2, 0)$	$\lambda_+ = -5.14\ldots \times 10^{-2}$ $\lambda_- = -5.828\ldots$
$(3.089\ldots, 0.2996\ldots)$	$\lambda_+ = 1.0399\ldots$ $\lambda_- = 8.0134\ldots \times 10^{-4}$
$(0.0527\ldots, -0.2996\ldots)$	$\lambda_+ = 1.0399\ldots$ $\lambda_- = 8.0134- \times 10^{-4}$

Figure 1 shows the basins of attraction for the two stable fixed points of T. The black regions consist of those initial conditions which, under iterations of T, converge to (π, J_0). The white are initial conditions which have converged to $(0, -J_0)$. In [14], this map was examined to determine what the consequences for prediction are due to the complicated intertwining of the basins of attraction. The question the authors of [14] asked was, given an initial condition (θ_0, x_0), do the orbits of $(\theta_0, x_0 + \epsilon)$ or $(\theta_0, x_0 \ \epsilon)$ also converge to the same fixed point as (θ_0, x_0), for some uncertainty ϵ? They found that the fraction of uncertain phase space volume was proportional to $\epsilon^{0.2}$, and concluded the system was very sensitive to small perturbations of the initial conditions. Using the capacity definition of dimension [15], they estimated the dimension of the basin boundary to be $d \approx 1.8$, and concluded that extraordinarily high accuracy of initial conditions may be necessary for reliable prediction of the eventual final state. We ask about the effect of NM on the map T in regards to final state sensitivity.

NEWTON'S METHOD

In order to solve to solve numerically for the unstable fixed points of T, it is necessary to employ a contracting map such as NM. Let $g(y) = y - T(y)$. Then the zeros of g are the fixed points (stable and unstable) of T. Newton's Method is used to find the zeros of g, hence the fixed points of T.

Let N denote the Newton map, defined by

$$N(y) = y - (g'(y))^{-1} g(y)$$

where $g'(y)$ denotes the 2×2 matrix of partial derivatives evaluated at y. It is well known [5] that if an initial point y_0 is sufficiently close to a zero of g, then the iterates given by $y_{n+1} = N(y_n)$ will converge to the that zero, which corresponds to a fixed point of T. This convergence now holds for both stable and unstable fixed points of T.

Here we determine how sensitive NM is to the basin boundary shown in Fig. 1. By observing the asymptotic behavior of iterates of N for a large set of initial conditions, we determine where the iterates of initial points not "sufficiently close" to a fixed point converge.

RESULTS

Figure 2 plots the basin of attraction of the fixed point (π, J_0) for the map N. The black dots correspond to initial points whose Newton iterates converge to (π, J_0). Comparison with Fig. 1 reveals that NM not only destroys the original basin structure (which is to be expected, since unstable fixed points of T correspond to stable fixed points of N), but has its own basin structure. The situation for NM is more complicated in that we are dealing with more than two stable fixed points for N. Indeed, a small but significant number of initial points get mapped to fixed points outside the current region of interest, $\theta \in (0, \pi)$.

To examine the structure of the Newton basin in detail, we look at the enlarged region (Figs. 3-5) enclosed by the rectangle with sides bordering on the unstable fixed point (3.089..., 0.29958...) and the stable fixed point $(\pi, 0.3)$ in the upper right-hand corner of Fig. 2. In Figs. 3-5, the black dots correspond to the Newton basins of attraction for the fixed points (3.089..., 0.2996...), $(\pi/2, 0)$, and (3.19..., 0.2996...) respectively. Approximately all the remaining initial conditions in the rectangle comprise the basin for $(\pi, 0.3)$.

Following [14], the sensitivity of the Newton map to some random ϵ imprecision in the initial condition was determined. Newton iterates for the initial points (θ_0, x_0), $(\theta_0 + \epsilon, x_0)$, and $(\theta_0 - \epsilon, x_0)$ were computed, where (θ_0, x_0) is an initial point chosen at random. If Newton iterates for at least one of the perturbed initial points converged to a different fixed point than the unperturbed initial point, then (θ_0, x_0) was said to be uncertain. One million random initial conditions were chosen, and the fraction, f, of uncertain initial points was computed. The value of ϵ was varied and the procedure repeated.

Figure 6 illustrates the scaling of f with ϵ on a log-log plot. Using a linear least squares fit, we see that the fraction of uncertain points is proportional to $\epsilon^{0.58}$. Thus, the capacity [15] of the Newton basin boundary is fractal with dimension $d \approx 1.42$. The fractal nature of the basin boundary can easily be seen in Fig. 7, which is an enlargement of the small rectangle at the top of Fig. 5.

CONCLUSIONS

We considered the result of applying a Newton map to a problem which exhibits complicated basin boundary structure. It was seen that although much of the original basin boundary is destroyed, the basin boundary structure for the Newton map possesses its own complex structure. The dimension (capacity) of the Newton basin was estimated to be approximately 1.42, revealing a fracted — like basin structure. Therefore, although the Newton map has only stable fixed points, global convergence is affected by the scaling law, $f \propto \epsilon^{a}$, where $a < 1$.

In the simple case of iterative mapping, ϵ has a lower bound dependent only upon machine precision and roundoff error. But, for maps constructed numerically from differential equations, ϵ will be bounded by the global truncation error of the particular numerical solver used. In the latter case, there could exist a significant number of uncertain initial conditions, adversely affecting the outcome of computer-assisted proofs using Newton iterates.

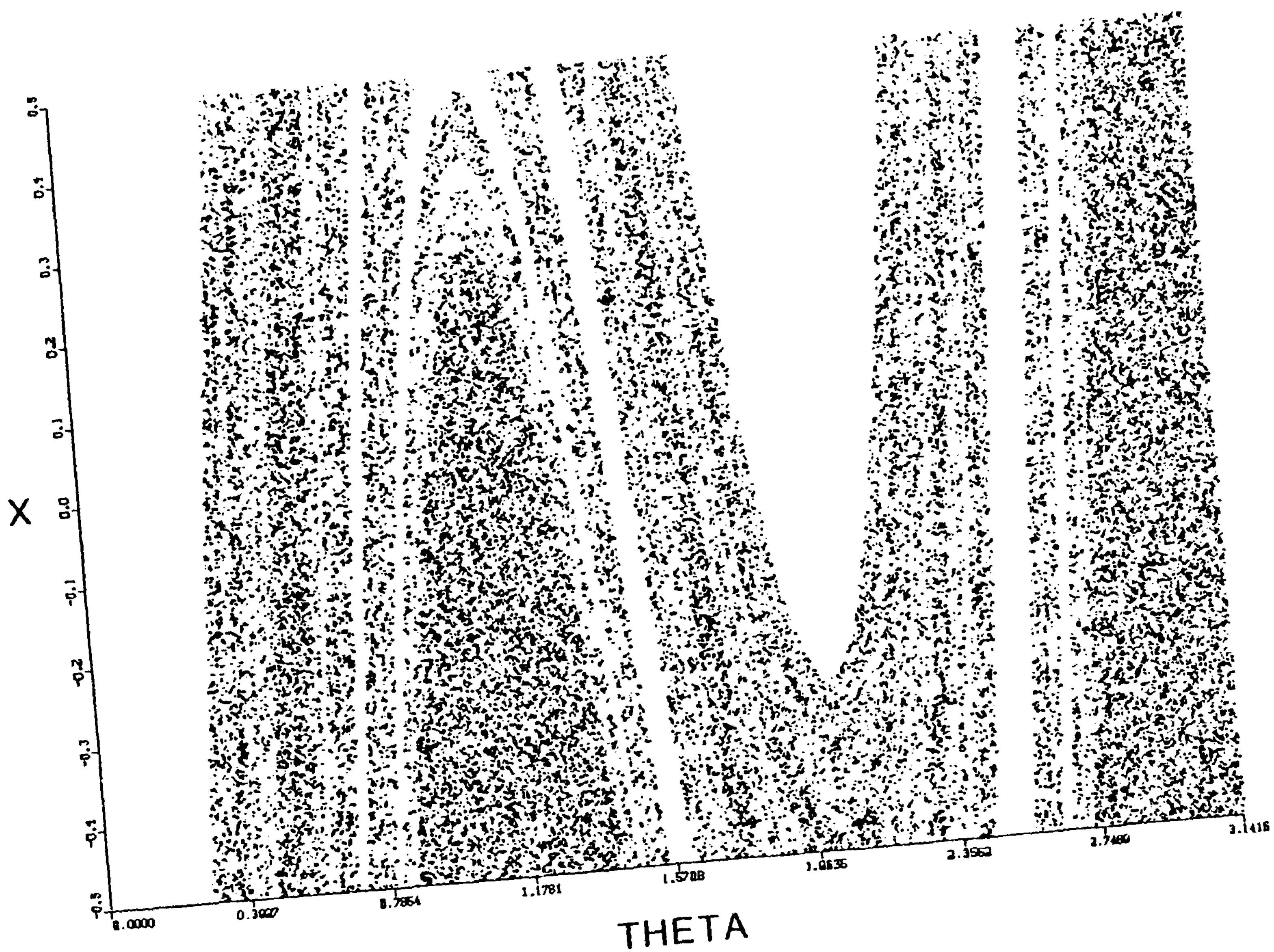

Fig. 1 — Basin of attraction for map T. Black dots are initial conditions that converge to that converge to $(\pi, 0.3)$, white areas $(0, -0.3)$. 60,000 random initial points were chosen parameter values: $A=1.32$, $B=0.9$, $J_0=0.3$.

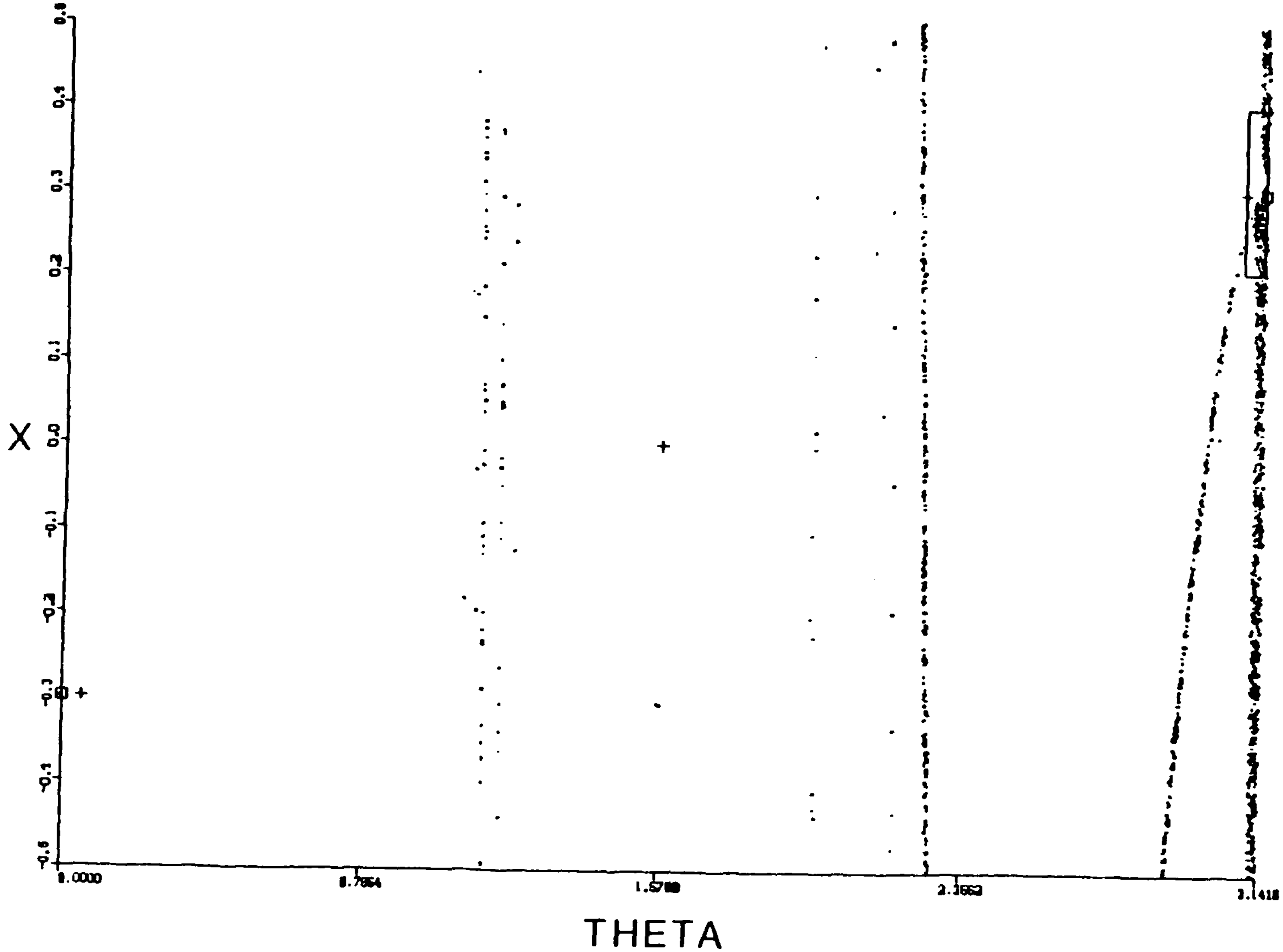

Fig. 2 — Basins for the Newton iterates of T. Black dots converge to $(\pi, 0.3)$. Squares denote the stable fixed points of T, and crosses denote unstable fixed points of T. Parameters: $A = 1.64$, $B = 0.9$, $J_0 = 3$. 100,00 initial random points were chosen. The outlined rectangular region is enlarged in Figs. 3,4,5.

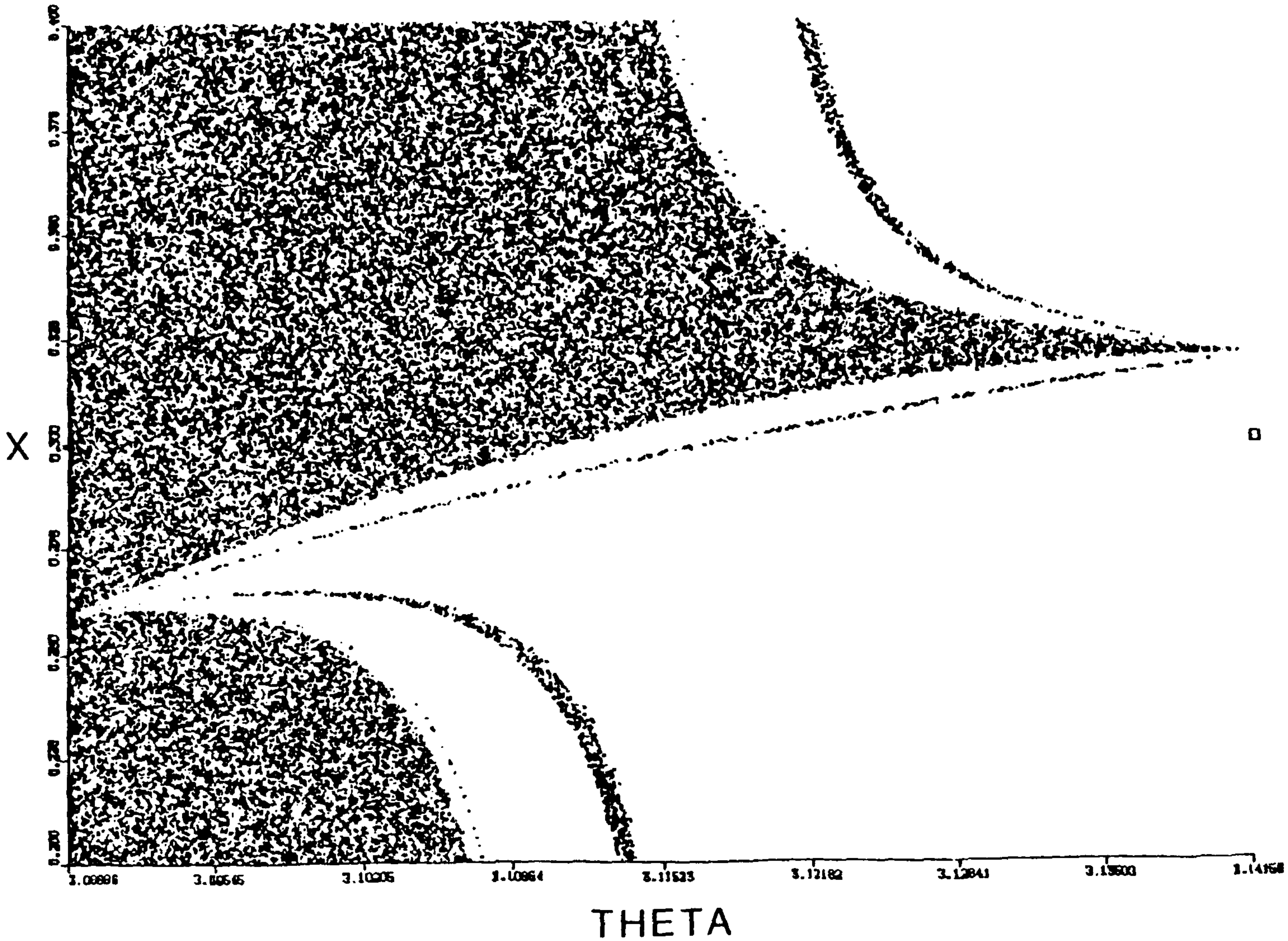

Fig 3 -- Newton basin for fixed point (3.089...,0.2996...) for the rectangular region shown in Fig. 2

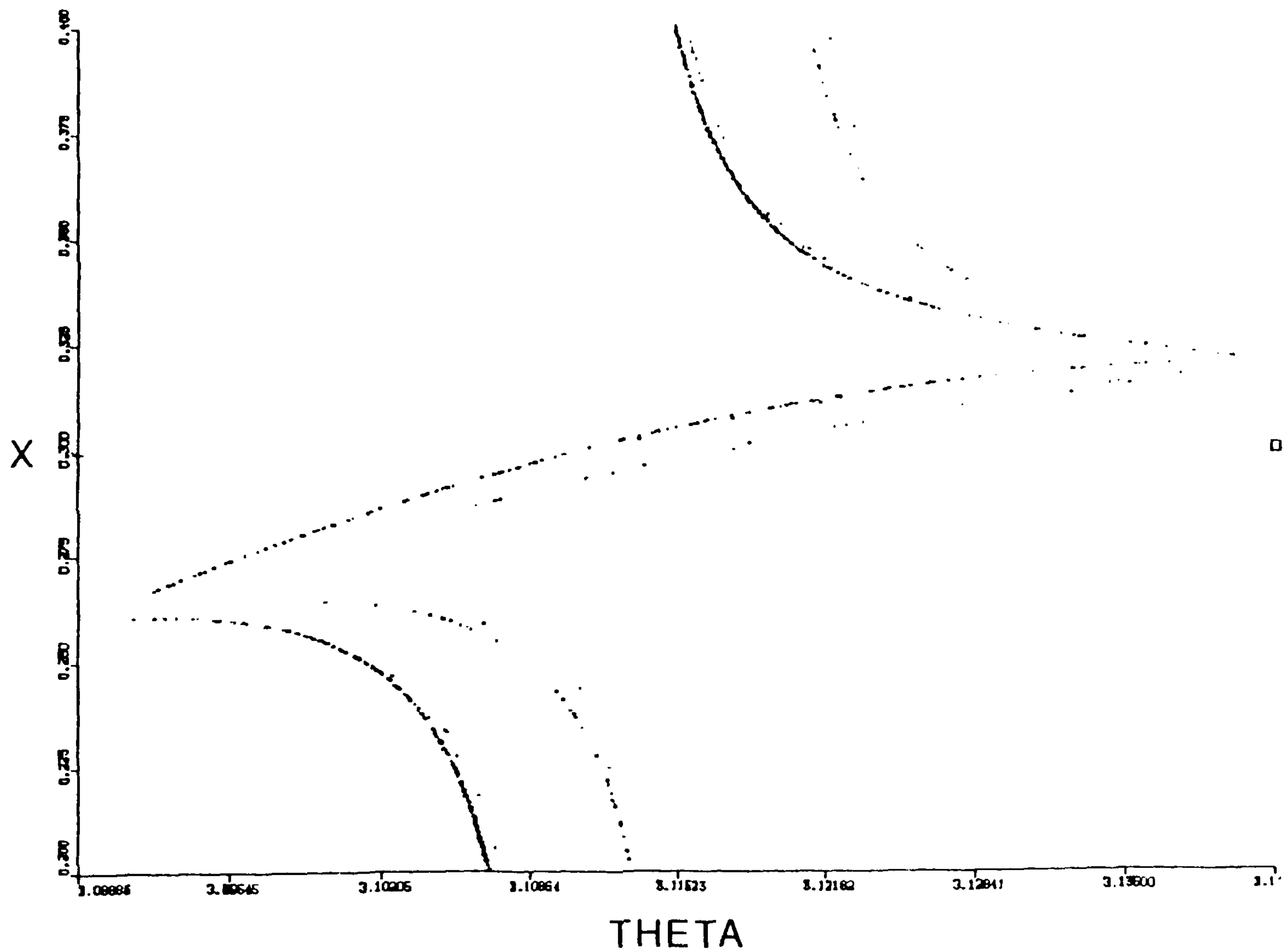

Fig. 4 — Newton basin for $(\pi/2,0)$

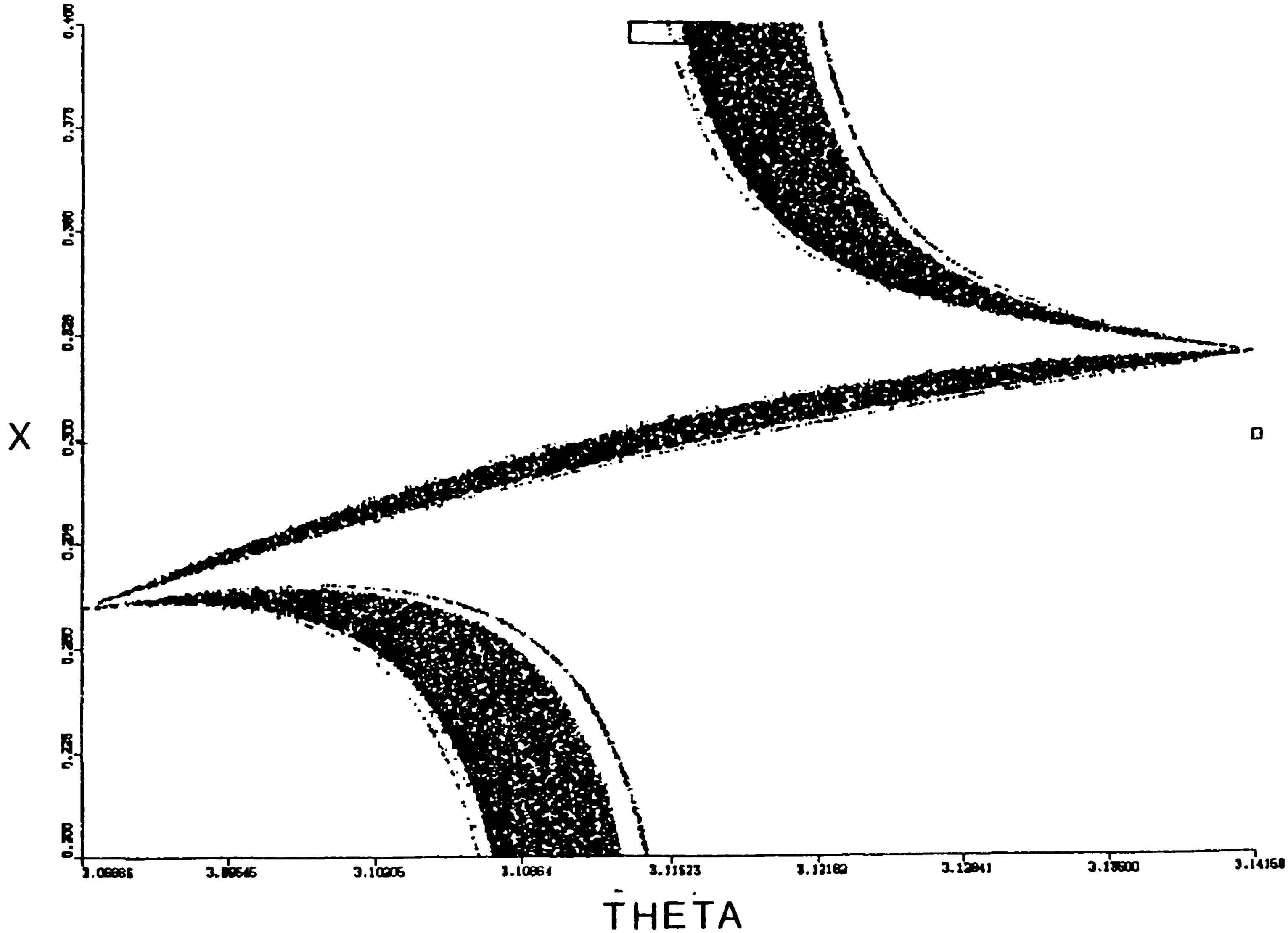

Fig. 5 — Newton basin for (3.19...,0.2996...). Rectangular region is enlarged in Fig. 7.

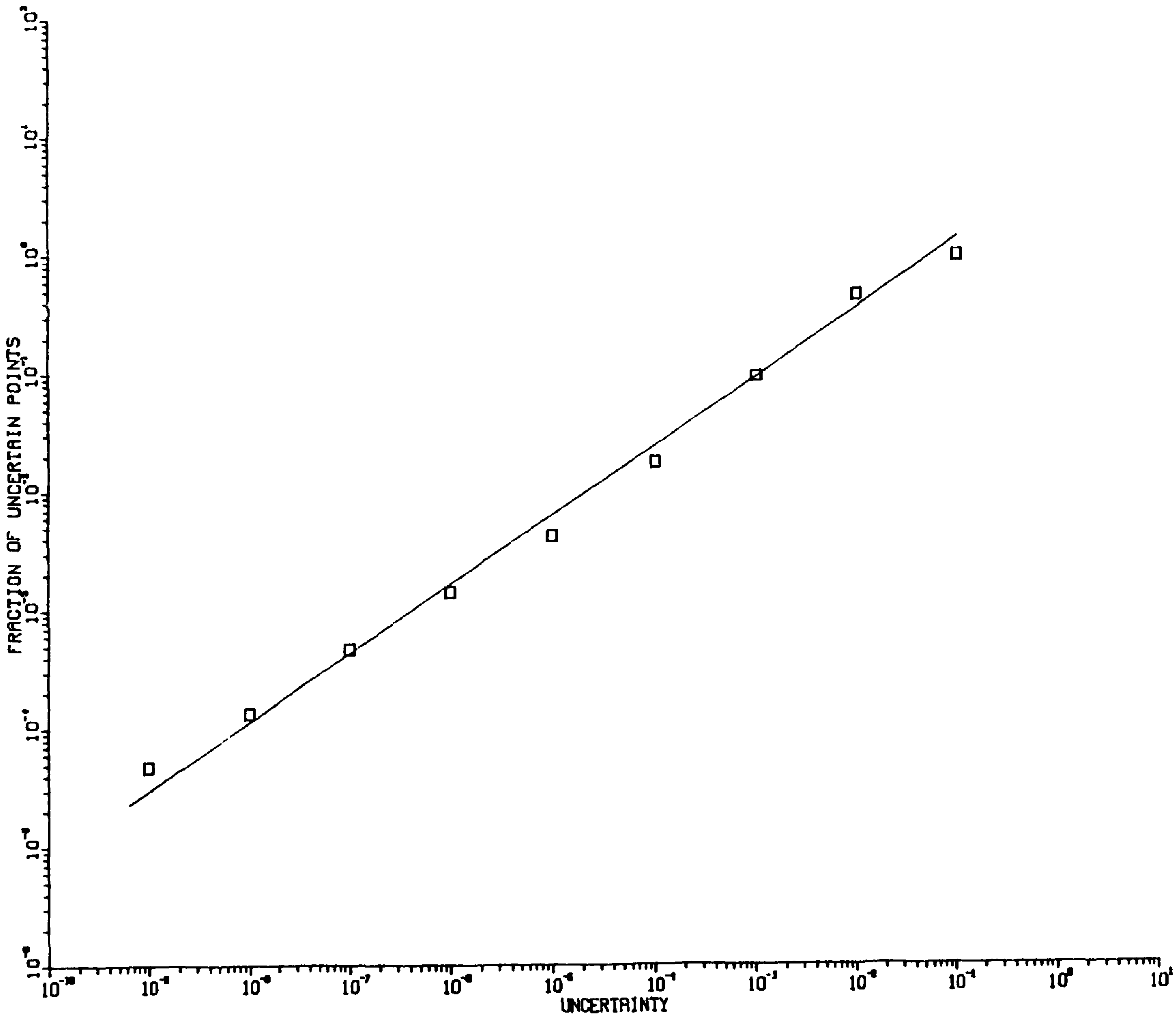

Fig. 6 — Fraction of uncertain points f vs. uncertainty in initial conditions, $\epsilon . f \propto \epsilon^{0.58}$. 10^6 initial points per value of ϵ were chosen for the region in Figs. 3,4,5.

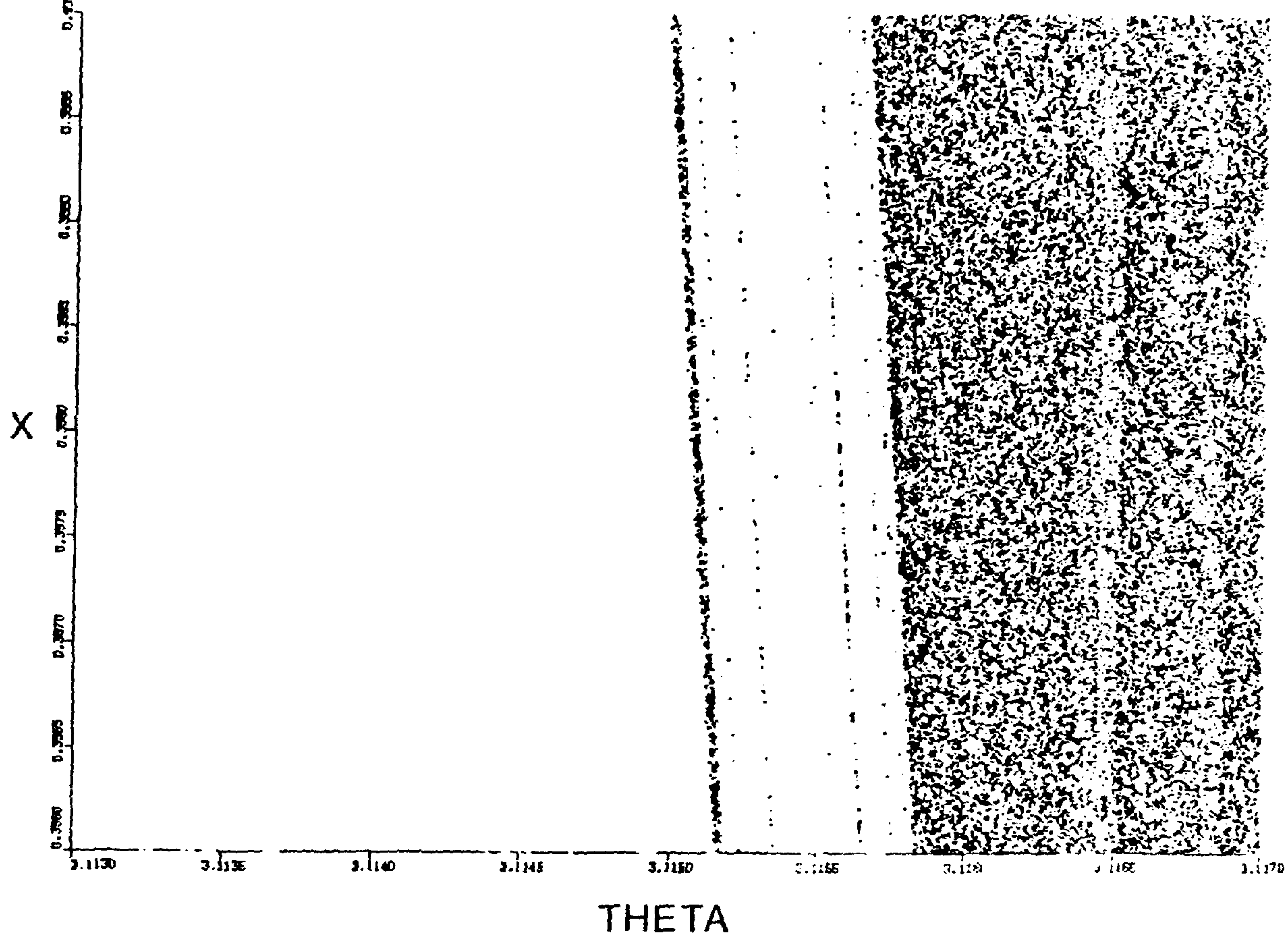

Fig. 7 — Enlargement of rectangle in Fig. 5 showing the fracted nature of the boundary

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